NOTIZEN

Eigenvalue Spectrum of the Radiation Field in the Flat Parallelepiped

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The eigenvalue spectrum of the ideal photon gas in the flat parallelepiped is discussed. For low frequencies the spectrum is that of the two dimensional isotropic space and tends with increasing ω to the spectrum of the 3-dimensional isotropic space.

Thermodynamic potentials and the thermodynamic properties of quantummechanical systems can be calculated if the distribution of eigenvalues (eigenvalue density) is known. The mean occupation number of the energy levels is determined by the quantum statistical behaviour of the particles resp. quasiparticles (for example, the mean coccupation number of photons is given in thermal equilibrium by the Bose-Einstein distribution).

For photons the energy ε is proportional to the frequency $\omega\,(\varepsilon = \hbar\,\omega)$ and $D(\omega)\,d\omega$ represents the number of eigenstates in the frequency interval $[\omega, \omega + d\omega]$. To obtain $D(\omega)$ we have to solve a counting problem: how many states $N(\omega)$ are lying below a fixed frequency ω? Once, $N(\omega)$ is known we find $D(\omega)$ via the relation $D(\omega) = dN(\omega)/d\omega$.

$N(\omega)$ for the infinite isotropic space is given by the phase space volume bounded by the energy surface $\varepsilon = \hbar\,\omega = \hbar\,c\,|\boldsymbol{k}|$ divided by $\hbar^3$. $\boldsymbol{k}$ denotes the wave number which is related to the momentum: $\boldsymbol{p} = \hbar\,\boldsymbol{k}$. Taking into consideration the twofold degeneracy of each state we obtain the well known result:

$$\lim_{V\to\infty} N(\omega)/V = \omega^3/3\,\pi^2\,c^3$$

and consequently for the density of states Weyl's term

$$\lim_{V\to\infty} D(\omega)/V = \omega^2/\pi^2\,c^3\,.$$

The limit must be taken with respect to the infinite isotropic space.

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A completely different behaviour is found in finite cavities with lossless walls: only discrete energy levels are possible. For example, we consider the cuboidal cavity. The eigenfrequencies ω_k are determined by three integers (n_1, n_2, n_3):

$$\omega_k = c\,\pi\,\left[\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2\right]^{1/2}$$

L_1, L_2, L_3 are the length of the cuboid edges. $N(\omega)$ is given by the number of allowed lattice points [2, 3] in the ellipsoid with the axis $L_1\,\omega/c\,\pi$, $L_2\,\omega/c\,\pi$, $L_3\,\omega/c\,\pi$:

$$N(\omega) = \sum_{\{\boldsymbol{k}\}} \Theta(\omega - \omega_k)\,. \tag{1}$$

$\{\boldsymbol{k}\}$ denotes the summation over all allowed modes [4]. $\Theta(x)$ is the Heaviside step function.

Formally, we may derive from (1) $D(\omega)$ as a series over δ-distributions [5, 6]:

$$D(\omega) = \sum_{\{\boldsymbol{k}\}} \delta(\omega - \omega_k)\,. \tag{2}$$

'Between' these two systems lies the system the spectrum of which we will discuss in this note: the flat parallelepiped $(L_1, L_2 \to \infty,\ L_3 = L,\ n_3 = n)$. The spectrum consists of a continuous and of a discrete part: the components of the wave number in x and y direction may have arbitrary values between 0 and infinite while k_z takes only discrete values (this is due to the boundary conditions of the electromagnetic field on the perfectly conducting plates). $D(\omega)$ is known [7, 8] and can be calculated from (2) replacing the summations over n_1 and n_2 by integrations.

Integration of $D(\omega)$ with respect to ω yields:

$$\lim_{L_1, L_2\to\infty} N(\omega)/V = \overline{N}(\omega) = \frac{1}{4\,\pi\,c^2\,L} \times \left\{\omega^2 + 2\sum_{n=1}^{\left[\frac{\omega L}{\pi c}\right]} [\omega^2 - (\pi\,c\,n/L)^2]\right\}. \tag{3}$$

$[\omega\,L/\pi\,c]$ is the greatest integer which is smaller or equal $\omega\,L/\pi\,c$.

The first term in (3) represents $\overline{N}$ for a two dim. isotropic infinite continuum ($\boldsymbol{k}$ lies in the xy-plane). If $\omega > \pi\,c/L$ additional modes with nonvanishing $\boldsymbol{k}$-component perpendicular to the plates appear. These modes are twofold degenerated [factor 2 in front of the series in (3)].

The increment in $N(\omega)$ due to a frequency increment $\Delta\omega$ may be split up in two parts: $\Delta\overline{N} = \Delta\overline{N}_{\parallel}$

$+ \Delta \overline{N}_{\mathrm{D}}$. $\Delta \overline{N}_{\|}$ represents the increase of the states of the 2-dim. continuous spectrum while $\Delta \overline{N}_{\mathrm{D}}$ describes the increase due to the combination of $\Delta\omega$ with the greatest k_z component with $c\,k_z \leqq \omega$. We consider two examples:

a) $\omega = \omega_1 + \Delta\omega$ where $\omega_1 = \pi c/L$ and $\Delta\omega \ll \omega_1$. We find: $\Delta \overline{N}_{\|} \approx (\omega_1^2/2\,\pi^2 c^3)\,\Delta\omega$ and $\Delta \overline{N}_{\mathrm{D}} \approx (\omega_1^2/\pi^2 c^3)\,\Delta\omega$.
$\Delta \overline{N}_{\mathrm{D}}$ may be interpreted as the difference of the number of states of a 3-dim. isotropic infinite system below the frequencies ω_1 and $\omega_1 + \Delta\omega$.

b) $\omega = \omega_3 + \Delta\omega$ where $\omega_3 = 3\pi c/L$ and $\Delta\omega \ll \pi c/L$. We find: $\Delta \overline{N}_{\|} \approx \frac{1}{6}(\omega_3^2/\pi^2 c^3)\,\Delta\omega$ and $\Delta \overline{N}_{\mathrm{D}} \approx (\omega_3^2/\pi^2 c^3)\,\Delta\omega$.

We see, that with increasing ω in $\Delta \overline{N}_{\|}$ the factor in front of $(\omega^2/\pi^2 c^3)\,\Delta\omega$ decreases while the factor in $\Delta \overline{N}_{\mathrm{D}}$ is constant $(=1)$. Generally, we can write:

$$\overline{N}\left(\omega = \frac{n\,\pi\,c}{L}\right) = \left[\frac{1}{4\,n} + \frac{1}{2\,n}\sum_{p=1}^{n-1}\left(1 - \frac{p^2}{n^2}\right)\right]\frac{\omega^3}{\pi^2 c^3}. \tag{4}$$

In comparison to the series in (4) the term $1/4\,n$ in (4) gets more and more unimportant with increasing n i.e. the two dimensional contributions in $\overline{N}$ may be neglected. In the limit $n \to \infty$ $(\omega \to \infty)$ we find:

$$\lim_{n\to\infty}\left(\frac{n-1}{2\,n} - \frac{1}{2\,n^3}\sum_{p=1}^{n-1}\right)p^2 = 1/3$$

and we obtain exactly the infinite isotropic space limit.

We conclude:

(i) The energy spectrum is continuous and there is no lowest mode with $\omega \neq 0$. Modes which have only a z-component of the wave vector dont exist (i.e. the weight of these modes is zero; this corresponds to the fact that for the cuboid no modes are allowed for which two of the integers n_1, n_2, n_3 are zero [4]).

(ii) Exceeding the frequencies $\omega_n = n\,\pi\,c/L$ new contributions appear which originate from the combination of the discrete k_z components with the $k_x - k_y$ continuum. This is revealed by jumps in the slope of $N(\omega)$ at the points ω_n.

(iii) The sum of the possible combinations of the discrete and continuous parts leads for great ω to a $N(\omega)$ which corresponds to the infinite isotropic space.

The properties of the spectrum we discussed in this note are revealed for instance in the properties of the blackbody radiation (energy, temporal coherence) in this geometry [9].

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